Isometries in the Plane

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A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is called an isometry if f preserves the euclidean distance, that is,

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We can define the product of two isometries as the composition of functions (performed from right to left) i.e

$$fg(P) = f(g(P))$$

It is clear that the product of two isometries is again an isometry. Also observe that the product is not commutative.

Examples

The following are typical examples of isometry.

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Example 0.4

Reflection in a line L passing through origin and making an inclination θ with positive $X-$ axis

$$s_\theta : (x, y) \to (x\cos 2\theta + y\sin 2\theta, x\sin 2\theta - y\cos 2\theta)$$

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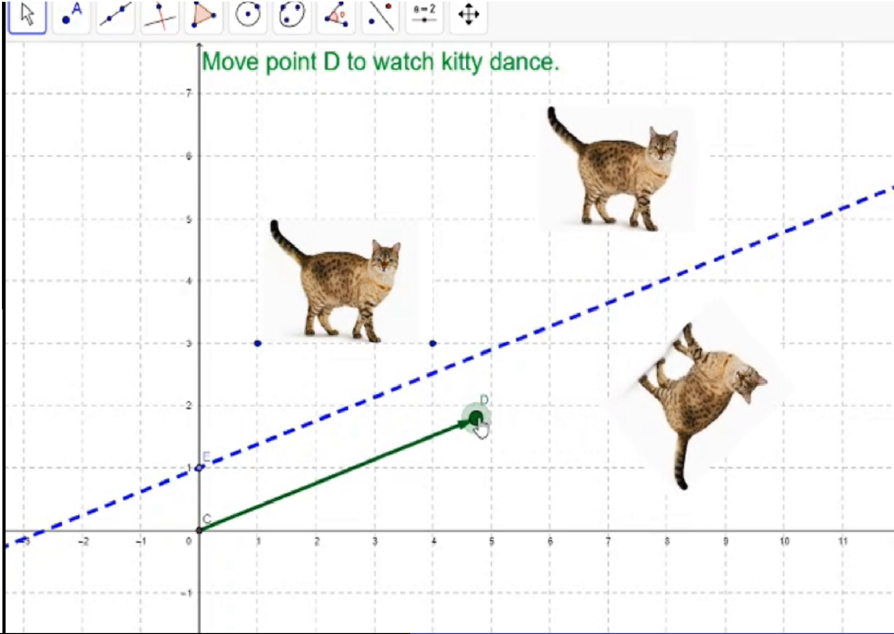
Example 0.8 (Half Turn)

Half turn is defined as the rotation by 180^o.

Glide Reflection



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- $S_{L_1}S_{L_2} = S_{L_3}S_{L_4}$ if the point of intersection and corresponding signed angles(or distance in case of paralle lines) are equal.
- The product of reflections in three parallel lines or in three concurrent lines is a reflection and a glide reflection otherwise.

Fixed Points

Definition 0.9

A point P is called fixed point of a function f, if $f(P) = P$.
Further, we say that f fixes a set S, if $f(S) = S$.
A function can have a set S fixed by f without having a fixed point.

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- A glide reflection has no fixed point.

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4. *An isometry that does not fix any point is a product of three reflections*

Theorem 0.11 (Three Reflections Theorem)

Every isometry in the plane is the product of at most three reflections

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- The determinant of the rotation matrix $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is 1 where as that of reflection matrix $B_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos\theta \end{pmatrix}$ is -1.

Table

Isometry	Sense	Fixed points	Fixed lines	Minimum Number of reflections
Reflection in line m	Opposite	Points on line m	m and all lines perpendicular to m	1, in line m
Identity map	Direct	All	All	2, in any one line
Rotation about O in angle $\theta \neq 180^\circ$	Direct	0(only)	None	2, in lines intersecting at O in angle $\theta/2$
Halfturn about O	Direct	0(only)	All lines through O	2, in lines perpendicular at O
Translation with vector $\overrightarrow{PQ}$	Direct	None	All lines parallel to $\overrightarrow{PQ}$	2, in lines perpendicular to $\overrightarrow{PQ}$ and half the length of $\overrightarrow{PQ}$ apart
Glide-reflection along m and in m	Opposite	None	m(only)	3, in m and two lines perpendicular to m

Group theoretic properties

- The set of isometries of the plane forms a group under composition denoted by $ISO(\mathbb{R}^2)$
- The set of translations and rotations (product of two reflections) is a normal subgroup of $ISO(\mathbb{R}^2)$ denoted by $ISO^+(\mathbb{R}^2)$
- A translation or a glide reflection generate an infinite subgroup of $ISO(\mathbb{R}^2)$. Thus a finite subgroup of $ISO(\mathbb{R}^2)$ contains only rotations and reflections.

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Theorem 0.12 (Leonardo da Vinci)

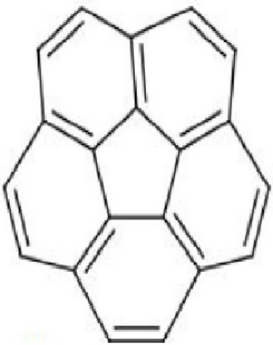
A finite subgroup of $ISO(\mathbb{R}^2)$ is either cyclic (in the case when it has only rotations) or dihedral (if it contains a reflection).

Theorem 0.13

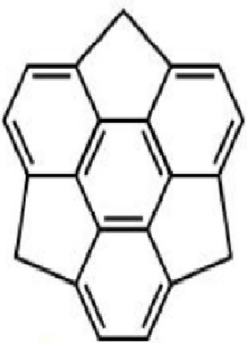
1. *The perpendicular bisectors of the sides of a triangle are concurrent (Circumcenter)*
2. *The angle bisectors of the angles of a triangle are concurrent (Incenter)*
3. *The medians of a triangle are concurrent (Centroid)*
4. *The altitudes of a triangle are concurrent (Orthocenter)*

An Application of Isometry to Chemistry

In computational Chemistry, one uses isometry to evaluate aromaticity or antiaromaticity of polycyclic aromatic allotropes like Corannulene, Sumanene etc which have polygons (like pentagon or hexagon) in their molecular structures.



Corannulene



Sumanene

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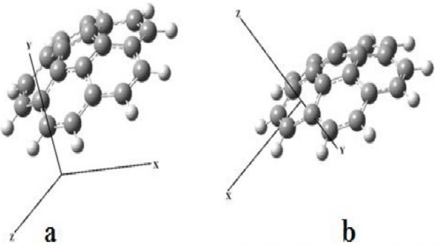
- For evaluation of aromaticity using NICS techniques one needs to estimate the components of nuclear magnetic shield tensors.
- One technique to estimate the components of nuclear magnetic shield tensors is to shift and rotate the molecule of Corannulene or Sumanene to centre of a polygon of such molecule.



a Corannulene molecule in original state

b Corannulene molecule after rotation and translation

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- Surfaces of constant curvature

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